Malaysian Journal of Mathematical Sciences 12(1): 121–141 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Some Solution of the Fractional Iterative Integro-Differential Equations

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Received: 14 May 2017
Accepted: 20 January 2018

ABSTRACT

In this article, we focus to some classes of fractional iterative integrodifferential equations. Firstly, we interested of the fractional iterative integrodifferential equations including derivatives and establish the existence and uniqueness solutions by using the non-expansive operators theory and fixed point theorems. The second studies, we concern of the system iterative integro-differential equations and show existence and uniqueness solutions by using the theorem of Banach fixed point and Schaefer fixed point theorem. In this study, we consider Riemann-Liouville and Caputo differential operator, further provide example as an application.

Keywords: Fractional; iterative; existence; fixed point theorem Schaefer's; non-expansive operator technique.

1. Introduction

During the past thirty years, there was a senior saucepan of studies in the area of iterative differential equations. On the other hand, the iterative differential equations of order fractional does not exceed ten years in terms of study and discussion. Those equations emerge in an enormous diversity of applications of scientific and technical, inclusive the modeling of issues from the naturalistic and social sciences like biological, economics, and physics (Loverro (2004), Miller and Ross (1993), Nieto and Rodríguez-López (2005), Salahshour et al. (2015), Srivastava and Agarwal (2013)).

A specific kind is appeared by the fractional differential equations with the affine amendment of the argument that can be retard fractional differential equations with a linear change of the argument. There were numerous outcomes with regard to these equations were presented in the papers (Caballero et al. (2007), Darwish (2008), Darwish and Ntouyas (2011), Kate and McLeod (1971), Ke (1994), Myshkis (1977), Norkin et al. (1973)).

Second kind of the type of differential equations of order fractional with amended arguments are the fractional differential equations with iteration like equation $D^{\beta}v(s) = v(v(s))$ where $0 < \beta < 1$. There were also quite a number of papers and the research dealt it (Agarwal et al. (2015), Atangana and Baleanu (2016), Atangana and Koca (2016), Cheng et al. (2002), Damag et al. (2016, 2017), Ibrahim et al. (2016, 2015), Lauran (2012, 2013), Wang et al. (2013), Zhang and Gong (2014), Zhang et al. (2015)).

In this article, to focus to two objectives for some classes of fractional iterative integrodifferential equations. Firstly, we interested of the fractional iterative integrodifferential equations including derivatives and prove the existence and uniqueness of solution by using the non-expansive operators theory and fixed point theorems. The second studies, we concern of the system iterative integro-differential equations and show existence and uniqueness solutions by using the theorem of Banach fixed point and Schaefer fixed point theorem. In this study, we consider Riemann-Liouville and Caputo differential operator, further provide example as an application.

2. Preliminaries

We recall several important of the definitions, notations, and theorems which are used in the paper (Berinde (2007), Ishikawa (1976), Miller and Ross (1993), Oregan (1995), Podlubny (1998), Samko et al. (1993)).

Definition 2.1. *(Agarwal et al. (2015), Podlubny (1998), Salahshour et al. (2015)) The integral operator is defined as*

$$I_a^{\alpha}\psi(s) = \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\psi(\beta)}{(s-\beta)^{1-\alpha}}d\beta \tag{1}$$

where $\alpha > 0$.

Definition 2.2. *(Podlubny (1998), Salahshour et al. (2015), Samko et al. (1993)) The fractional differentiation operator (**Caputo**) is defined as*

$$D_0^{\alpha}\psi(s) = \frac{1}{\Gamma(\iota-\alpha)}\int_0^s \frac{\psi^{(\iota)}(\beta)}{(s-\beta)^{\alpha-\iota+1}}d\beta \tag{2}$$
$$(\iota - 1) \leq \alpha < \iota,$$

where ι *is a whole number and* $\alpha > 0$ *is a real number.*

Definition 2.3. *(Miller and Ross (1993), Podlubny (1998), Srivastava and Agarwal (2013)) The fractional differentiation operator (**Riemann-Liouville**) is defined as*

$$D_0^{\alpha}\psi(s) = \frac{1}{\Gamma(\iota-\alpha)}[\frac{d}{ds}]^{\iota}\int_0^s \frac{\psi(\beta)}{(s-\beta)^{\alpha-\iota+1}}d\beta \tag{3}$$
$$(\iota - 1) \leq \alpha < \iota,$$

where ι *is a whole number and* α *is a real number.*

Definition 2.4. *(Berinde (2007), Ishikawa (1976), Zhang et al. (2015))* A *is a space of normed linear,* Q *is a convex and* $Q \subset A$ *and* H *is a self-mapping defined by* $H : Q \to Q$*. In view of* $v_0 \in Q$ *and* $\xi \in [0, 1]$ *is a real number,* v_i *is a sequence defined by the formula*

$$v_{i+1} = (1 - \xi_i)v_i + \xi_i H v_i, \quad i = 0, 1, 2, \ldots$$

is generally called Mann iteration.

Definition 2.5. *(Berinde (2007), Oregan (1995))* (Z, d) *is a space of metric and* H *is mapping defined by* $H : Z \to Z$ *said to be an* η*-contraction if there is* $\eta \in [0, 1)$ *such that*

$$d(Hz, Hw) \leq \eta d(z, w), \forall z, w \in Z.$$

When $\eta = 1$*, therefore the mapping* H *is said to be non-expansive.*

Definition 2.6. *(Berinde (2007)) A is a space of normed linear and $Q \subset A$ is convex and H is a self-mapping introduced by $H : Q \to Q$. In view of $v_0 \in Q$ and ξ is the real numbers in $[0, 1]$, v_i is a sequence defined by the formula*

$$v_{i+1} = (1 - \xi)v_i + \xi H v_i, \quad i = 0, 1, \ldots .$$

In general referred Krasnoselskij iteration or Krasnoselskij-Mann iteration .

Theorem 2.1. : *A is space of Banach, Q sub set A, and let H be a non expansive mapping defined by $H : Q \to Q$. If process Mann iteration s_i fulfills the postulates:*

(i) $s_i \in Q$ *for each positive integer* i,

(ii) $0 \leq \xi_i \leq b < 1$, *for each positive integer* m,

(iii) $\sum_{i=0}^{\infty} \xi_i = \infty$. *Whether s_i is bounded, next $s_i - Hs_i \to 0$ as $i \to \infty$.*

Corollary 2.1. : *A is a real normed space and $Q \subset A$ is a closed bounded, convex, and H is a non expansive mapping defined by$H : Q \to Q$. If $I - H$ maps closed bounded subsets of A into closed subsets of A and s_i is Mann iteration, with ξ_i is fulfilled postulates $(i) - (iii)$ in theorem 2.7, then s_i is a strongly converges to a fixed point of $H \in Q$.*

Lemma 2.1. *Given $\ell_1([a, b], R)$, then $\forall s \in [a, b]$, have*

$$I^{\alpha}.I^{\gamma} g(s) = I^{\alpha+\gamma} g(s), \; for \; \alpha, \gamma > 0.$$

$$D^{\gamma} I^{\gamma} g(s) = g(s), \; for \; \gamma > 0.$$

$$D^{\alpha} I^{\gamma} g(s) = I^{\gamma-\alpha} g(s), for \; \gamma > \alpha > 0.$$

Lemma 2.2. *For $m - 1 < \beta < m$, where $m \in N*$, the general solution of the equation $D^{\beta} y(s) = 0$ is given by*

$$y(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{m-1} t^{m-1}, \tag{4}$$

where $c_j \in R, j = 0, 1, 2, \ldots, m - 1$.

Lemma 2.3. *A is Banach space and $Q \subset A$ is a nonempty, compact, and convex. Then any non-expansive mapping $H : Q \to Q$ has at least a fixed point.*

3. Main Results

In this section, to focus for two aims for establish the existence solutions of some classes of fractional iterative integrodifferential equations. Firstly,

we interested of the fractional iterative integro-differential equations including derivatives as:

$$D^{\beta}v(s) = g\Big(s, v(v(s)), v(v'(s)), \int_{s_0}^{s} K(s,r).v(v(r))dr\Big), \tag{5}$$

with

$$v(s_0 = v_0$$

where s_0, v_0 in $I = [a,b]$, $g : I \times I \times I \times I \to I$ and $k : I \times I \to I$ are continuous functions and using theory of the non-expansive operators and theorems of fixed point to prove. The second studies, the system iterative integro-differential equations are form:-

$$\begin{cases} D^{\beta_1}v(s) = \phi_1(s)g_1(s, v(s), v(v(s), z(s), z(z(s))) \\ + \int_0^s \frac{(s-r)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} g_1(r, v(r), z(r), v(v(r)), z(z(r)))dr \\ D^{\beta_2}z(s) = \phi_2(s)g_2(s, v(s), v(v(s), z(s), z(z(s))) \\ + \int_0^s \frac{(s-r)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} g_2(r, v(r), z(r), v(v(r)), z(z(r)))dr \\ v(0) = a, z(0) = b, s \in [0,1] \end{cases} \tag{6}$$

where D^{β_1}, D^{β_2} denote the Caputo fractional derivatives, $0 < \beta_j < 1$, $j = 1,2$, α_1 and α_2 are non-negative real numbers, ϕ_1, ϕ_2 are two continuous functions, $a > 0, b > 0$, g_1 and g_2 are two functions to be specified later, and using the theorem of Banach fixed point and theorem of Schaefer fixed point to prove and we consider Riemann-Liouville and Caputo differential operator, further provide example as an application.

3.1 The Fractional Iterative Integro-differential Equations Including Derivatives

We prove the existence and uniqueness of the solution of Eq.(5) and the non-expansive operators theory and theorems of fixed point to use.

In the following assumptions relating to our further discussion.

Assumptions 3.1. *The following assumptions are:-*
(A_1) *There is* $\ell_1 > 0$ *so that*

$$|g(\chi, \Upsilon_1, , w_1, z_1) - g(\chi, \Upsilon_2, w_2, z_2)| \leq \ell_1[|\Upsilon_1 - \Upsilon_2| + |w_1 - w_2| + |z_1 - z_2|] \tag{7}$$

for every $\chi, \Upsilon_m, w_m, z_m \in I, m = 1, 2,$
(A_2) *if* ℓ *there is a constant such that* $|v(s_1) - v(s_2)| \leq \ell \cdot \frac{|s_1 - s_2|^\beta}{\Gamma(\beta+1)}$, *subsequently*

$$M = max\left\{|g(\chi, \Upsilon, w, z)| : (\chi, \Upsilon, w, z) \in I \times I \times I \times I\right\} \leq \frac{\ell}{2},$$

(A_3) *One of these situations are satisfied:*

$$(i)\, \frac{M.T^\beta}{\Gamma(\beta+1)}.M_{s_0} \leq M_{v_0}, \ \ where\ T = max\left\{a, b\right\},\ and\ M_{v_0} = max\left\{v_0 - a, b - v_0\right\};$$

$$(ii)\, s_0 = 0,\ M\frac{(T)^\beta}{\Gamma(\beta+1)} \leq b - v_0,\ g(\chi, \Upsilon, w, z) \geq 0,\ \ \forall(\chi, \Upsilon, w, z) \in I,$$

$$(iii)\, s_0 = b,\ M\frac{(T)^\beta}{\Gamma(\beta+1)} \leq v_0 - a,\ g(\chi, \Upsilon, w, z) \geq 0,\ \ \forall(\chi, \Upsilon, w, z) \in I,$$

(A_4) $v_0 \leq \frac{\rho \ell_2}{2}, 0 \neq s_0 \in I, \rho \in (0, 1).$
(A_5) *if* ℓ *there is a constant so that* $|v(s_1) - v(s_2)| \leq \ell \cdot \frac{|s_1 - s_2|^\beta}{\Gamma(\beta+1)}$, *therefore* $M = min\left\{\frac{\rho}{2}, \frac{\ell}{2}\right\}$

Let $C(I, I)$ be the Banach space of each continuous functions from $I \to I$ given with the norm $\|v\| = sup\{\, v(s) \,:\, s \in I\}$, $M_s = max\left\{s - a, b - s\right\}$ and

$$C_{\ell,\beta} = \left\{v \in C(I, I) : |v(s_1) - v(s_2)| \leq \ell \cdot \frac{|s_1 - s_2|^\beta}{\Gamma(\beta+1)},\ \forall s_1, s_1 \in I\right\}, \qquad (8)$$

where $\ell > 0$ is given. It is evident which $C_{\ell,\beta} \neq \emptyset$ is subset from $(C[I], \|.\|)$, convex, and compact.

Now, start to study some theorems about the fractional iterative integro-differential equations including derivatives to study the existence solutions:

Theorem 3.1. : *Assume which assumptions* $(A_1) - (A_3)$ *are fulfilled and*

$$2\ell_1 M_{s_0}\Big[1 + \frac{T^\beta}{\Gamma(\beta+1)} K_T\Big](\ell + 1) \leq 1 \qquad (9)$$

In which $K_T = sup\{\, K(r, s) : a \leq r \leq s \leq b\}$. *Then, the question* (5) *which has at least one solution in* $C_{\ell,\beta}$, *which can be approximated by Krasnoselskii of iteration*

$$v_{m+1}(u) = (1 - \eta)v_m(u)$$

$$+\eta v_0+\eta\int_{s0}^{u}\frac{(u-\mu)^{\beta}}{\Gamma(\beta+1)}g\Big(\mu,v_m(v_m(\mu)),v_m(v_m'(\mu)),\int_{s_0}^{\mu}K(\mu,r)v_m(v_m(r))dr\Big)d\mu,$$

where u in I, $m\geq 1$, $u>\mu$, $\eta\in(0,1)$ and $v_1,v_1'\in C_{\ell,\beta}$ is arbitrary.

Proof. Let $G:C_{\ell,\beta}\to C[I]$ be the integral operator defined by

$$(Gv)(u)=v_0+\int_{s0}^{u}\frac{(u-\mu)^{\beta}}{\Gamma(\beta+1)}g\Big(\mu,v(v(\mu)),v(v'(\mu)),\int_{s_0}^{\mu}K(\mu,r)v(v(r))dr\Big)d\mu,\ \ u\ in\ I,\ u>\mu$$

$v=Gv$ is a solution of the Eq.(5) for any fixed point. Firstly, prove which C_ℓ is an invariant set with respect to G(i.e. $G(C_{\ell,\beta})\subset C_{\ell,\beta}$).
Using assumption $(A_3)(i)$, obtain

$$\begin{aligned}|(Gv)(u)|&\leq|v_0|+\Big|\int_{s0}^{u}\frac{(u-\mu)^{\beta}}{\Gamma(\beta+1)}g\Big(\mu,v(v(\mu)),v(v'(\mu)),\int_{s_0}^{\mu}K(\mu,r)v(v(r))dr\Big)d\mu\Big|\\&\leq v_0+M\frac{(s_0-u)^{\beta}}{\Gamma(\beta+1)}M_{s_0}\\&\leq v_0+M_{v_0}=v_0+b-v_0\\&\leq b\end{aligned}$$

and

$$\begin{aligned}|(Gv)(u)|&\geq|v_0|-\Big|\int_{s0}^{u}\frac{(u-\mu)^{\beta}}{\Gamma(\beta+1)}g\Big(\mu,v(v(\mu)),v(v'(\mu)),\int_{s_0}^{\mu}K(\mu,r)v(v(r))dr\Big)d\mu\Big|\\&\geq v_0-M\frac{(s_0-u)^{\beta}}{\Gamma(\beta+1)}M_{s_0}\\&\geq v_0-M_{v_0}=v_0-v_0+a\\&\geq a\end{aligned}$$

Consequently, $Gv\in C_{\ell,\beta}$ for every $v\in C_{\ell,\beta}$ and $s\in I$.
Likewise, the result is obtained using the assumption $(A_3)(ii)$ and $(A_3)(iii)$.
Using the assumption (A_2) for each $s_1,s_2\in I$, get

$$\begin{aligned}|(Gv)(u_1)-(Gv)(u_2)|\leq&\Big|\int_{s0}^{u_1}\frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)}g\Big(\mu,v(v(\mu)),v(v'(\mu)),\int_{s_0}^{\mu}K(\mu,r)v(v(r))dr\Big)d\mu\\&-\int_{s0}^{u_2}\frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)}g\Big(\mu,v(v(\mu)),v(v'(\mu)),\int_{s_0}^{\mu}K(\mu,r)v(v(r))dr\Big)d\mu\Big|\end{aligned}$$

$$\leq M.\frac{|u_1^{\beta}-u_2^{\beta}|+2|u_1-u_2|^{\beta}}{\Gamma(\beta+1)}$$

$$\leq 2M.\frac{|u_1-u_2|^{\beta}}{\Gamma(\beta+1)} \leq \ell.\frac{|u_1-u_2|^{\beta}}{\Gamma(\beta+1)}$$

Subsequently, Gv in $C_{\ell,\beta}$ for every v in $C_{\ell,\beta}$. Then $G : C_{\ell,\beta} \to C_{\ell,\beta}$ (i.e. G is a self-mapping).
Using the assumption (A_1) for each $v, w \in C_{\ell,\beta}$, and $s \in I$, get

$$|(Gv)(u)-(Gw)(u)| = \Big|\int_{s0}^{u}\frac{(u-\mu)^{\beta}}{\Gamma(\beta+1)}g\Big(\mu, v(v(\mu)), v(v'(\mu)), \int_{s_0}^{\mu}K(\mu,r)v(v(r))dr\Big)d\mu$$

$$-\int_{s0}^{u}\frac{(u-\mu)^{\beta}}{\Gamma(\beta+1)}g\Big(\mu, w(w(\mu)), w(w'(\mu)), \int_{s_0}^{\mu}K(\mu,r)w(w(r))dr\Big)d\mu\Big|$$

$$\leq \int_{s_0}^{u}\left|\frac{(u-\mu)^{\beta}}{\Gamma(\beta+1)}\Big[g\Big(\mu, v(v(\mu)), v(v'(\mu)), \int_{s_0}^{\mu}K(\mu,r)v(v(r))dr\Big)\right.$$

$$\left.-g\Big(\mu, w(w(\mu)), w(w'(\mu)), \int_{s_0}^{\mu}K(\mu,r)w(w(r))dr\Big)\Big]d\mu\right|$$

$$\leq \ell_1 M_{s_0}\Big[1+\frac{T^{\beta}}{\Gamma(\beta+1)}K_T\Big]\int_{s_0}^{u}\Big[|v(v(\mu))-w(w(\mu))|+|v(v'(\mu))-w(w'(\mu))|\Big]d\mu$$

$$\leq \ell_1 M_{s_0}\Big[1+\frac{T^{\beta}}{\Gamma(\beta+1)}K_T\Big]\int_{s_0}^{u}\Big[|v(v(\mu))-v(w(\mu))+v(w(\mu))-w(w(\mu))|$$

$$+|v(v'(\mu))-v(w'(\mu))+v(w'(\mu))-w(w'(\mu))|\Big]d\mu$$

$$\leq \ell_1 M_{s_0}\Big[1+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}K_T\Big]\int_{s_0}^{u}\Big[\ell|v(\mu)-w(\mu)|+|v(w(\mu))-w(w(\mu))|$$

$$+\ell|v'(\mu)-w'(\mu)|+|v(w'(\mu))-w(w'(\mu))|\Big]d\mu$$

$$\leq 2\ell_1 M_{s_0}\Big[1+\frac{T^{\beta}}{\Gamma(\beta+1)}K_T\Big](\ell+1)\|v-w\|$$

in which $K_T = sup\{K(r,s) : a \leq r \leq s \leq b\}$. Presently, by taking the maximum in last inequality, obtain

$$\|(Gv)-(Gw)\| \leq 2\ell_1 M_{s_0}\Big[1+\frac{T^{\beta}}{\Gamma(\beta+1)}K_T\Big](\ell+1)\|v-w\|.$$

In view of equation(9)
(a) If $2\ell_1 M_{s_0}\Big[1+\frac{T^{\beta}}{\Gamma(\beta+1)}K_T\Big](\ell+1) < 1$, thus G is a contraction mapping

and therefore by the theorem of Banach fixed point equation(5) has a unique solution.
(b) If $2\ell_1 M_{s_0}\Big[1+\frac{T^\beta}{\Gamma(\beta+1)}K_T\Big](\ell+1)=1$, thus G is nonexpansive mapping and therefore it is continuous. So lemma 11 means that equation(5) has a solution in $C_{\ell,\beta}$.

Lastly, by applying theorem 2.7 or corollary 2.8, we get the second part of the theorem.
Then prove the result of equation(5) in a subset of $C_{\ell,\beta}$ introduced by

$$C_{\ell,\beta,\rho}=\left\{v\in C_{\ell,\beta}: v(s)\leq \frac{\rho s^\beta}{\Gamma(\beta+1)},\ \ for\ all\ \ s\in I\right\},\ \ \rho\in(0,1).$$

It is clear which $C_{\ell,\beta,\rho}$ is convex, not empty, and compact subset in $C(I)$.$\diamond$

Theorem 3.2. : *Suppose which assumptions (A_1), $(A_3)-(A_5)$ are achieved. If*

$$\boldsymbol{L}=max_{s\in I}\left\{2\frac{\ell_1}{\gamma}\Big[1+\frac{T^\beta}{\Gamma(\beta+1)}K_T\Big]\Big(\ell\left|(1-e^{-\gamma(s-s_0)})\right|+\frac{1}{\rho}\left|e^{\gamma(\rho-1)s}-e^{\gamma(\rho s_0-s)}\right|\Big)\right\}\leq 1 \tag{10}$$

Thus, there is at least one solution of the equation (5) in $C_{\ell,\beta,\rho}$ that can be approximated by the iteration of Krasnoselskij

$$v_{m+1}(u)=(1-\eta)v_m(u)+\eta v_0$$
$$+\eta\int_{s0}^{u}\frac{(u-\mu)^\beta}{\Gamma(\beta+1)}g\Big(\mu, v_m(v_m(\mu)), v_m(v_m(\mu)), v_m(v_m'(\mu)), \int_{s_0}^{\mu}K(\mu,r)v_m(v_m(r))dr\Big)d\mu,$$

in which u in I, $m\geq 1$, $u>\mu$, η in $(0,1)$ and $v_1, v_1'\in C_{\ell,\rho,\beta}$ is arbitrary.

Proof. Let $C_(I)$ be Banach space with the norm given by the Bieleckis formula

$$\|v\|_B=max_{s\in I}\left\{\|v(s)\|\,e^{-\gamma(s-s_0)},\ \gamma>0,\ \ s>s_0\right\}$$

Let G be introduced as proof of theorem 3.1.2, by hypothesis (A_1), $(A_3)-(A_5)$, it is appropriate to show which, if $v\in C_{\ell,\rho,\beta}$, therefore $G(v)\in C_{\ell,\rho,\beta}$.
For $v\in C_{\ell,\rho,\beta}$, and $s\in I$, have

$$Gv(s)\leq v_0+M\frac{s^\beta}{\Gamma(\beta+1)}$$

$$= v_0 + M\frac{(s^\beta - s_0^\beta) + s_0^\beta}{\Gamma(\beta+1)}$$

$$\leq \frac{s_0^\beta}{2\Gamma(\beta+1)} + \frac{s^\beta}{2\Gamma(\beta+1)} - \frac{s_0^\beta}{2\Gamma(\beta+1)} + \frac{s_0^\beta}{2\Gamma(\beta+1)}$$

$$\leq \frac{s^\beta}{\Gamma(\beta+1)}, s > s_0$$

This proves which $G(v) \in C_{\ell,\rho,\beta}$ and thus $C_{\ell,\rho,\beta}$ is invariant under G.
Presently, for each $v, w \in C_{\ell,\beta}$ and $s \in I$, get

$$|(Gv)(u) - (Gw)(u)| = \Bigg| \int_{s0}^{u} \frac{(u-\mu)^\beta}{\Gamma(\beta+1)} g\Big(\mu, v(v(\mu)), v(v'(\mu)), \int_{s_0}^{\mu} K(\mu,r)v(v(r))dr\Big) d\mu$$

$$- \int_{s0}^{u} \frac{(u-\mu)^\beta}{\Gamma(\beta+1)} g\Big(\mu, w(w(\mu)), w(w'(\mu)), \int_{s_0}^{\mu} K(\mu,r)w(w(r))dr\Big) d\mu \Bigg|$$

$$\leq \ell_1 \Big[1 + \frac{T^\beta}{\Gamma(\beta+1)} K_T\Big] \Bigg| \int_{s_0}^{u} \Big(\ell\,|v(\mu) - w(\mu)| + |v(w(\mu)) - w(w(\mu))|$$

$$+\ell\,|v'(\mu) - w'(\mu)| + |v(w'(\mu)) - w(w'(\mu))|\,\Big) d\mu$$

$$\leq \ell_1 \Big[1 + \frac{T^\beta}{\Gamma(\beta+1)} K_T\Big] \Big(\Big| \int_{s_0}^{u} (2\ell e^{\gamma(\mu - s_0)}) d\mu \Big| + \Big| \int_{s_0}^{u} (2e^{\gamma(\mu\rho - s_0)}) d\mu \Big| \Big) \|v - w\|_B$$

$$\leq 2\ell_1 \Big[1 + \frac{T^\beta}{\Gamma(\beta+1)} K_T\Big] \Big(\Big| \frac{\ell}{\gamma}(e^{\gamma(s-s_0)} - 1) \Big| + \frac{1}{\rho\gamma} \Big| e^{\gamma(s\rho - s_0)} - e^{\gamma(s_0\rho - s_0)} \Big| \Big) \|v - w\|_B .$$

Thus,

$$|(Gv)(u) - (Gw)(u)|\, e^{-\gamma(s-s_0)} \leq 2\frac{\ell_1}{\gamma} \Big[1 + \frac{T^\beta}{\Gamma(\beta+1)} K_T\Big] \Big(\ell \Big|(1 - e^{-\gamma(s-s_0)}) \Big|$$

$$+ \frac{1}{\rho} \Big| e^{\gamma(\rho-1)s} - e^{\gamma(\rho s_0 - s)} \Big| \Big) \|v - w\|_B\, \Gamma$$

Presently, taking a maximum in the final inequality gives

$$\|(Gv)(u) - (Gw)(u)\|_B \leq \mathbf{L}\, \|v - w\|_B .$$

In view of equation (10)
(a) If $\mathbf{L} < 1$, thus G is a contraction mapping and therefore by the theorem of Banach fixed point equation (5) has a unique solution.
(b) If $\mathbf{L} = 1$, thus G is non-expansive mapping and therefore it is continuous. So lemma 11 means that equation (5) has a solution in $C_{\ell,\rho,\beta}$.

Lastly, applying theorem 2.7 or corollary 2.8, we get the second part of the theorem.$\diamond$

3.2 System of fractional iterative integro-differential equations

We considered the existence and uniqueness of the solution of Eq.(3.3). The theorem of Banach fixed point and Schaefer fixed point theorem are used to prove.

Now, we start our result in this section:

Lemma 3.1. *Let $g \in C([0,1], R)$. The solution of the problem*

$$D^{\beta}y(t) = (\phi g)(t) + \int_0^t \frac{(t-r)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} g(r)dr, 0 < \beta < 1, \alpha_1 > 0 \tag{11}$$

subject to the boundary condition,

$$y(0) = y_0^*$$

is given by

$$y(t) = \int_0^t \frac{(t-r)^{\beta-1}}{\Gamma(\beta)} (\phi g)(r)dr + \int_0^t \frac{(t-r)^{\beta+\alpha_1-1}}{\Gamma(\alpha_1+\beta)} g(r)dr + y_0^* \tag{12}$$

Proof. Setting

$$Z(t) = y(t) - I^{\beta}(\phi g)(t) - I^{\alpha_1+\beta}g(t), \tag{13}$$

we obtain

$$D^{\beta}Z(t) = D^{\beta}y(t) - D^{\beta}I^{\beta}(\phi g)(t) - D^{\alpha_1+\beta g(t),}(14)$$

Therefore, by lemma 2.9,

$$D^{\beta}Z(t) = D^{\beta}y(t) - (\phi g)(t) - I^{\alpha_1}g(t), \tag{15}$$

Hence, (11) is equivalent to $D^{\beta}z(t) = 0$. Finally, from lemma 2.10, we get that $z(t)$ is constant, (i.e., $z(t) = z(0) = y(0) = y_0^*$), and the proof of lemma is achieved

Assumptions 3.2. *The assumptions are:-*
(B_1) *if ℓ there is a constant so that $|v(s_1) - v(s_2)| \leq \ell \cdot \frac{|s_1-s_2|^{\beta}}{\Gamma(\beta+1)}$, therefore $M = \frac{\ell}{2}$*
(B_2) *There are non-negative real numbers m_j, n_j, $(j = 1, 2)$, so that $\forall s \in [0,1]$ and $(v_1, z_1), (v_2, z_2) \in R^2$, have*

$$g_1(s, v_2, z_2, v_2(v_2), z_2(z_2)) - g_1(s, v_1, z_1, v_1(v_1), z_1(z_1)) \leq m_1(\ell+1)|v_2-v_1| + m_2(\ell+1)|z_2-z_1|,$$

$g_2(s, v_2, z_2, v_2(v_2), z_2(z_2)) - g_2(s, v_1, z_1, v_1(v_1), z_1(z_1)) \leq n_1(\ell+1)|v_2 - v_1| + n_2(\ell+1)|z_2 - z_1|,$

(B_3) *The functions* g_1 *and* $g_2 : [0,1] \times R^4 \to R$*, are continuous*
(B_4) *There are two positive numbers* ℓ_1 *and* ℓ_2*, so that*

$$g_1(s, v, z, v(v), z(z)) \leq \ell_1, \; g_2(s, v, z, v(v), z(z)) \leq \ell_2, \; s \in [0,1], \; (v,z) \in R^2.$$

Theorem 3.3. *Presume that* (B_2) *achieves. Setting*

$$M_1 := \frac{||\phi_1||_\infty}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_1 + \alpha_1 + 1)},$$

$$M_2 := \frac{||\phi_2||_\infty}{\Gamma(\beta_2 + 1)} + \frac{1}{\Gamma(\beta_2 + \alpha_2 + 1)},$$

$M = max\,\{M_1, M_2\}$*. Then if*

$$(M_1 + M_2)(m_1 + m_2 + n_1 + n_2)(\ell + 1) < 1, \tag{16}$$

the fractional system (3.3) *has exactly one solution on* $[0,1]$*.*

Proof. Let us consider

$$Y := C([0,1], R).$$

This space, equipped with the norm $||.||_Y = ||.||_\infty$ introduced by

$$||g||_\infty = \sup\left\{|g(Y)|, \; X \; in \; [0,1]\right\},$$

is a Banach space. Also, the product space $(Y \times Y, ||(v,z)||_{Y \times Y})$ is a Banach space, with norm $(v,z)_{Y\times Y} = ||v||_Y + ||z||_Y$.
Consider now the operator $\Psi : Y \times Y \to Y \times Y$, introduced by

$$\Psi(v,z)(s) = (\Psi_1(v,z)(s), \Psi_2(v,z)(s)), \tag{17}$$

where,

$$\Psi_1(v,z)(s) = \int_0^s \frac{(s-r)^{\beta_1 - 1}}{\Gamma(\beta_1)} \phi_1(r) g_1(r, v(r), z(r), v(v(r)), z(z(r)))dr$$
$$+ \int_0^s \frac{(s-r)^{\beta_1 + \alpha_1 - 1}}{\Gamma(\beta_1 + \alpha_1)} g_1(r, v(r), z(r), v(v(r)), z(z(r)))dr + a \tag{18}$$

and

$$\Psi_2(v,z)(s) = \int_0^s \frac{(s-r)^{\beta_2 - 1}}{\Gamma(\beta_2)} \phi_2(r) g_2(r, v(r), z(r), v(v(r)), z(z(r)))dr$$

$$+\int_0^s \frac{(s-r)^{\beta_2+\alpha_2-1}}{\Gamma(\beta_2+\alpha_2)} g_2(r, v(r), z(r), v(v(r)), z(z(r)))dr + b \tag{19}$$

We shall show that T is a contraction. Let $(v_1, z_1), (v_2, z_2) \in Y \times Y$. Therefore, for each $s \in [0, 1]$, get

$$|\Psi_1(v_2, z_2)(s) - \Psi_1(v_1, z_1)(s)|$$

$$\leq \Big(\int_0^s \frac{(s-r)^{\beta_1-1}}{\Gamma(\beta_1)} \sup_{0\leq r\leq 1} |\phi_1(r)|dr + \int_0^s \frac{(s-r)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)} dr\Big)$$

$$\times \sup_{0\leq r\leq 1} |g_1(r, v_2(r), z_2(r), v_2(v_2(r)), z_2(z_2(r))) - g_1(r, v_1(r), z_1(r), v_1(v_1(r)), z_1(z_1(r)))|.$$

$\forall s \in [0, 1]$, we obtain

$$\begin{aligned} |\Psi_1(v_2, z_2)(s) - \Psi_1(v_1, z_1)(s)| &\leq \left(\tfrac{||\phi_1||_\infty}{\Gamma(\beta_1+1)} + \tfrac{1}{\Gamma(\beta_1+\alpha_1+1)}\right) \\ \times \sup\nolimits_{0\leq r\leq 1} |g_1(r, v_2(r), z_2(r), v_2(v_2(r)), z_2(z_2(r)))&-g_1(r, v_1(r), z_1(r), v_1(v_1(r)), z_1(z_1(r)))| \end{aligned} \tag{20}$$

Using (B_2), we can write:

$$|\Psi_1(v_2, z_2)(s) - \Psi_1(v_1, z_1)(s)| \leq M_1(m_1(\ell+1)(v_2-v_1) + m_2(\ell+1)(z_2-z_1)) \tag{21}$$

Then,

$$|\Psi_1(v_2, z_2)(s) - \Psi_1(v_1, z_1)(s)| \leq M_1(m_1+m_2)(\ell+1)\Big(||v_2-v_1||_Y + ||z_2-z_1||_Y\Big) \tag{22}$$

Thus,

$$|\Psi_1(v_2, z_2)(s) - \Psi_1(v_1, z_1)(s)| \leq M_1(m_1+m_2)(\ell+1)||v_2-v_1, ||z_2-z_1||_{Y\times Y} \tag{23}$$

With the same arguments, as above, we obtain

$$|\Psi_2(v_2, z_2)(s) - \Psi_2(v_1, z_1)(s)| \leq M_2(n_1+n_2)(\ell+1)||v_2-v_1, ||z_2-z_1||_{Y\times Y}. \tag{24}$$

Finally, using (23) and (24), conclude that

$$||\Psi(v_2, z_2)(s) - \Psi(v_1, z_1)(s)||_{Y\times Y}$$

$$\leq (M_1+M_2)(\ell+1)(m_1+m_2+n_1+n_2)||v_2-v_1, ||z_2-z_1||_{Y\times Y}. \tag{25}$$

From (16), we come to the conclusion that T is a contraction mapping. Therefore, by Banach's fixed-point theorem, there is a unique fixed point, that is a solution of (3.3).$\diamond$

Theorem 3.4. *Presume that* (B_3) *and* (B_4) *are fulfilled. Then the problem* (3.3) *has at least one solution on [0, 1].*

Proof. Firstly, to prove that the operator T is completely continuous. (Observe that T is continuous on $Y \times Y$ to the continuity of g_1 and g_2).
Stride 1:- Let us take $\xi > 0$ and $A_\xi := \{(v, z) \in Y \times Y; ||(v, z)||_{Y \times Y} \leq \xi\}$, and presume that (B_4) achieves. Then, for $(v, z) \in A_\xi$, have

$$\begin{aligned} |T_1(v,z)(s)| \leq & \frac{s^{\beta_1} \sup_{0\leq s\leq 1} \cdot|\phi_1(s)|}{\Gamma(\beta_1+1)} \sup_{0\leq s\leq 1} \cdot g_1(s, v(s), z(s), v(v(s)), z(z(s))) \\ & + \frac{s^{\beta_1+\alpha_1}}{\Gamma(\beta_1+\alpha_1+1)} \sup_{0\leq s\leq 1} \cdot g_1(s, v(s), z(s), v(v(s)), z(z(s))) + a \end{aligned} \tag{26}$$

$\forall s \in [0, 1]$, and by (B_4) , we get

$$||T_1(v,z)(s)||_{Y\times Y} \leq \ell_1(\ell+1)M_1 + a < +\infty. \tag{27}$$

Also, have

$$||T_2(v,z)(s)||_{Y\times Y} \leq \ell_2 M_2(\ell+1) + b < +\infty. \tag{28}$$

Therefore, by (27) and (28),

$$||T(v,z)||_{Y\times Y}$$

is bounded by C, where

$$C := (\ell_1.M_1 + \ell_2.M_2)((\ell+1)) + a + b. \tag{29}$$

Stride 2:- The equi-continuity of T: Let $s_1, s_2 \in [0, 1], s_1 < s$ and $(v, z) \in A_\xi$. Since $0 < \beta_1 < 1$, therefore, we can write

$$\begin{aligned} |T_1(v,z)(s_1) - T_1(v,z)(s_2)| \leq \Big| & \int_0^{s_2} |\frac{(s_2-r)^{\beta_1-1}}{\Gamma(\beta_1)} \phi_1(r) g_1(r, v(r), z(r), v(v(r)), z(z(r)))dr \\ & - \int_0^{s_1} |\frac{(s_1-r)^{\beta_1-1}}{\Gamma(\beta_1)} \phi_1(r) g_1(r, v(r), z(r), v(v(r)), z(z(r)))dr \\ & + \int_0^{s_2} |\frac{(s_2-r)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)} g_1(r, v(r), z(r), v(v(r)), z(z(r)))dr \\ & - \int_0^{s_1} |\frac{(s_1-r)^{\beta_1+\alpha_1-1}}{\Gamma(\beta_1+\alpha_1)} g_1(r, v(r), z(r), v(v(r)), z(z(r)))dr \end{aligned} \tag{30}$$

Using (B_4), we can write

$$\begin{aligned} |T_1(v,z)(s_2) - T_1(v,z)(s_1)| \leq & \frac{\ell_1(\ell+1)||\phi_1(r)||_\infty \left(s_2^{\beta_1} - s_1^{\beta_1} + (s_2-s_1)^{\beta_1}\right)}{\Gamma(\beta_1+1)} \\ & + \frac{\ell_1(2\ell+2)\left(s_2^{\beta_1+\alpha_1} - s_1^{\beta_1+\alpha_1} + (s_2-s_1)^{\beta_1+\alpha_1}\right)}{\Gamma(\beta_1+\alpha_1+1)} \end{aligned} \tag{31}$$

Similarly, we can have

$$|T_2(v,z)(s_2) - T_2(v,z)(s_1)| \leq \frac{(\ell+1)\ell_1||\phi_2(r)||_\infty\left(s_2^{\beta_2} - s_1^{\beta_2} + (s_2-s_1)^{\beta_2}\right)}{\Gamma(\beta_2+1)} + \frac{\ell_1(2\ell+2)\left(s_2^{\beta_2+\alpha_2} - s_1^{\beta_2+\alpha_2} + (s_2-s_1)^{\beta_2+\alpha_2}\right)}{\Gamma(\beta_2+\alpha_2+1)} \tag{32}$$

From (31) and (32), yields

$$\begin{aligned}|T(v,z)(s_2) - T(v,z)(s_1)| \leq{}& \frac{\ell_1(\ell+1)||\phi_1(r)||_\infty\left(s_2^{\beta_1} - s_1^{\beta_1} + (s_2-s_1)^{\beta_1}\right)}{\Gamma(\beta_1+1)} \\ &+ \frac{\ell_1(\ell+1)\left(s_2^{\beta_1+\alpha_1} - s_1^{\beta_1+\alpha_1} + (s_2-s_1)^{\beta_1+\alpha_1}\right)}{\Gamma(\beta_1+\alpha_1+1)} \\ &+ \frac{\ell_1(\ell+1)||\phi_2(r)||_\infty\left(s_2^{\beta_2} - s_1^{\beta_2} + (s_2-s_1)^{\beta_2}\right)}{\Gamma(\beta_2+1)} \\ &+ \frac{\ell_1(\ell+1)\left(s_2^{\beta_2+\alpha_2} - s_1^{\beta_2+\alpha_2} + (s_2-s_1)^{\beta_2+\alpha_2}\right)}{\Gamma(\beta_2+\alpha_2+1)}\end{aligned} \tag{33}$$

As $s_2 \leftrightarrow s_1$, the right side of (33) tends to zero. Therefore, following strides $1, 2$ and by the ArzelAscoli theorem, deduce that T is completely continuous. Then, to consider the set

$$\Omega = \{(v,z) \in Y \times Y/(v,z) = \eta T(v,z), 0 < \eta < 1, (34)$$

and we prove that it is bounded. Let $(v,z) \in \Omega$, therefore $(v,z) = \eta T(v,z)$, for some $0 < \eta < 1$. Subsequently, for $s \in [0,1]$, have

$$v(s) = \eta T_1(v,z)(s), z(s) = \eta T_2(v,z)(s). \tag{35}$$

consequently,

$$||(v(s), z(s)||_{Y\times Y} = \eta||T(v,z)||_{Y\times Y}. \tag{36}$$

From (B_4), we obtain

$$||(v(s), z(s)||_{Y\times Y} \leq \eta C, \tag{37}$$

in which C is introduced by (29). We get that Ω is bounded.
In conclusion of the fixed points Schaefer theorem, it follows that T has at least one fixed point, which is a solution of (3.3).

4. Examples

In this section, we show some example to explain our theorems.

Example 4.1. *Consider the initial value problem linked to fractional iterative contain derivatives and integral equation following*

$$\begin{gathered} D^{\frac{1}{2}}v(s) = -\tfrac{1}{4} + \tfrac{1}{7}(v(v(s)) + v(v'(s)) + \tfrac{1}{16}\int_0^s \tfrac{1}{(2+s)^2}(v(v(r))dr \\ v(0) = \tfrac{1}{3}, v'(0) = \tfrac{1}{3} \end{gathered} \tag{38}$$

where s in $[0,1]$, and v in $C^{\ell,\frac{1}{2}}([0,1]\times[0,1])$.
Equation(38) is of the form equation(5)

$$g(s, v(v(s)), v(v'(s)), K_1v(v(s))) = -\frac{1}{4} + \frac{1}{7}(v(v(s)) + v(v'(s)) + \frac{1}{16}K_1v(v(s))$$

in which

$$K_1v(v(s)) = \int_0^s \frac{1}{(2+s)^2}(v(v(r)))dr$$

for any $v_1, v_2, v_1', v_2' \in C^{1,\frac{1}{2}}([0,1]\times[0,1]), ands \in I$ that, in given our notes, that

$$\begin{aligned} &|g(s, v_1(v_1(s)), v_1(v_1'(s)), K_1v_1(v_1(s))) - g(s, v_2(v_2(s)), v_2(v_2'(s)), K_1v_2(v_2(s)))| \\ &\le \frac{1}{7}\Big(|v_1(v_1(s)) - v_2(v_2(s))| + |v_1(v_1'(s)) - v_2(v_2'(s))|\Big) + \frac{1}{16}\Big(|K_1v_1(v_1(s)) - K_1v_2(v_2(s))|\Big) \\ &\le \frac{1}{7}\Big(|v_1(v_1(s)) - v_2(v_2(s))| + |v_1(v_1'(s)) - v_2(v_2'(s))|\Big) + \frac{1}{16}\Big(|K_1v_1(v_1(s)) - K_1v_2(v_2(s))|\Big) \\ &\le \frac{1}{7}\Big(|v_1(v_1(s)) - v_2(v_2(s))| + |v_1(v_1'(s)) - v_2(v_2'(s))|\Big) + |K_1v_1(v_1(s)) - K_1v_2(v_2(s))| \end{aligned} \tag{39}$$

Thus $\ell_1 = \frac{1}{7}$, and $M_{v_0} = max\{v_0 - a, b - v_0\} \to M_{\frac{1}{3}} = max\left\{\frac{1}{3}, \frac{2}{3}\right\} = \frac{2}{3}$.
The concerning with the solution of $v \in C^{\ell,\frac{1}{2}}([0,1]\times[0,1])$ of equation(38) belonging to the set

$$C_{\ell,\frac{1}{2}} = \left\{ v : |v(u_1) - v(u_2)| \le \ell.\frac{|u_1-u_2|^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)} \quad \forall u_1, u_2 \in [0,1] \right\} \text{ with } \ell = 1$$

$$C_{1,\frac{1}{2}} = \left\{ v : |v(u_1) - v(u_2)| \le \frac{1}{0.886}|u_1-u_2|^{\frac{1}{2}}, \ \forall u_1, u_2 \in [0,1] \right\} \tag{40}$$

Presently, $M \le \frac{\ell}{2} = \frac{1}{2}$, $M_{s_0} = max\{s_0 - a, b - s_0\} \to M_{s_o} = max\{0, 1\} = 1$, such that $\frac{M.T^\beta}{\Gamma(\beta+1)}.M_{s_0} = (\frac{1}{2})(\frac{1}{0.886})(1) = 0.5643 < M_{v_0} = \frac{2}{3}$
Hence

$$2\ell_1 M_{s_0}\Big[1 + \frac{T^\beta}{\Gamma(\beta+1)}K_T\Big](\ell+1) = 0.28571\Big[1 + \frac{1}{0.3544}\Big](\frac{3}{2}) = 0.23398 < 1 \tag{41}$$

From Equations $(38)-(41)$ *observe which among all the hypothesis of theorem* 3.1.3 *are achieved, and therefore the initial value equation* (38) *has a unique solution in* $C_{1,\frac{1}{2}}$ *which can be approximated by iteration of Krasnoselskii*

$$v_{m+1}(s) = (1-\eta)v_m(s) + \eta v_0 + \eta\int_0^s \frac{(s-\mu)^{\beta-1}}{\Gamma(\beta)}\Big(-\frac{1}{4}+\frac{1}{7}(v(v(s)) + v(v'(s))$$

$$+\frac{1}{16}\int_0^s \frac{1}{(2+s)^2}(v(v(r))dr\Big) \quad s\in I,$$

in which $m\geq 1$, $s>\mu$, $\eta\in(0,1)$ *and* $v_1, v'(1)\in C_{1,\frac{1}{2}}$ *is arbitrary.*$\diamond$

Example 4.2. *Consider the following fractional differential system:*

$$\begin{cases} D^{0.5}v(s) = \frac{e^{-s}}{32\sqrt{1+s}}\Big(\frac{\sin(v(v(s))+z(z(s)))}{18(\ln(s+1)+1)}+1\Big) \\ \quad +\int_0^s \frac{(s-r)^{2.5}}{\Gamma(3.5)}\Big(\frac{\sin(v(v(r))+z(z(r)))}{18(\ln(r+1)+1)}+1\Big)dr, \\ D^{0.5}z(s) = \frac{e^{-s^2}}{32\sqrt{1+s^2}}\Big(\frac{\sin(v(v(s)))+\sin(z(z(s)))}{16(e^{s^2}+1)}\Big) \\ \quad +\int_0^s \frac{(s-r)^{1.5}}{\Gamma(2.5)}\Big(\frac{\sin(v(v(r)))+\sin(z(z(r)))}{16(e^{r^2}+1)}\Big)dr, \\ v(0)=\sqrt{3}, z(0)=\sqrt{2}, s\in[0,1] \end{cases} \quad (42)$$

where, $\beta_{1,2}=0.5, \alpha_1=3.5,$ *and* $\alpha_2=2.5, a=\sqrt{3}, b=\sqrt{2}$, $g_1(s,v,z,v(v),z(z)) = \frac{\sin(v(v(s))+z(z(s)))}{18(\ln(s+1)+1)}+1, g_2(s,v,z,v(v),z(z)) = \frac{\sin(v(v(s)))+\sin(z(z(s)))}{16(e^{s^2}+1)}$, $\phi_1(s) = \frac{e^{-s}}{32\sqrt{1+s}}$ *and* $\phi_2(s) = \frac{e^{-s^2}}{32\sqrt{1+s^2}}$. *For* $(v_1,z_1),(v_2,z_2)\in R^2$, $s\in[0,1]$, *have*

$$|g_1(s,v_2,z_2,v_2(v_2),z_2(z_2)) - g_1(s,v_1,z_1,v_1(v_1),z_1(z_1))|$$

$$\leq \frac{1}{18}(|v_2-v_1|+|z_2-z_1|)$$

$$|g_2(s,v_2,z_2,v_2(v_2),z_2(z_2)) - g_2(s,v_1,z_1,v_1(v_1),z_1(z_1))|$$

$$\leq \frac{1}{16}(|v_2-v_1|+|z_2-z_1|)$$

Then, $M_1 = 0.076$, $M_2 = 0.201$, $M = max\{M_1,M_2\} = 0.201$. *Therefore,* $M=\frac{\ell}{2}\Leftrightarrow \ell = 0.402, m_1=m_2=0.07789$, $n_1,n_2 = 0.0876$ *Hence,* $(M_2+M_1)(\ell+1)(m_2+m_1+n_2+n_1) = 0.128537<1$

The conditions of the theorem 3.2.3 *achieved. Then, the problem* (42) *has a unique solution on* $[0,1]$.$\diamond$

Example 4.3. *Consider the following fractional differential system:*

$$\begin{cases} D^{\frac{3}{4}}v(s) = \cosh(1-\pi^2 s)\cos(v(v(s)+z(z(s))+ln(s+4) \\ \quad +\int_0^s \frac{(s-r)^{\sqrt{11}-1}}{\Gamma(\sqrt{11})}\Big(\cos(v(v(r)+z(z(r))+\ln(r+4)\Big)dr, \\ \quad D^{\frac{5}{7}}z(s) = \sinh(1-\pi s^2)se^{(-v(v(s))-z(z(s)))} \\ \quad +\int_0^s \frac{(s-r)^{\sqrt{7}-1}}{\Gamma(\sqrt{7})}\Big(re^{(-v(v(r))-z(z(r)))}\Big)dr, \\ \quad v(0)=2, z(0)=\sqrt{5}, s\in[0,1] \end{cases} \tag{43}$$

where, $\beta_1 = \frac{3}{4}, \beta_2 = \frac{5}{7}, \alpha_1 = \sqrt{11}$, *and* $\alpha_2 = \sqrt{7}, a = 2, b = \sqrt{5}, \forall s \in [0,1]$.*Also have* $\phi_1(s) = \cosh(1-\pi^2 s)$ *and* $\phi_2(s) = \sinh(1-\pi s^2)$. *For each* $(v,z) \in R^2$, *have*

$$g_1(s,v,z,v(v),z(z)) = \cos(v(v(s)+z(z(s))+\ln(s+4)$$

$$g_2(s,v,z,v(v),z(z)) = se^{(-v(v(s))-z(z(s)))}$$

It is obvious that g_1 *and* g_2 *are bounded and continuous functions. Under the conditions of theorem 3.2.4, the problem* (43) *has at least one solution in* $[0,1]$.$\diamond$

Competing interests The authors declare that they have no competing interests.

Authors′ contributions Both authors jointly worked on deriving the results and approved the final manuscript.

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